

Alternative subtraction scheme using Nagy-Soper dipoles

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Loopfest 10, Stony Brook, 23.6.10

1 NLO calculations - pole structure and treatments

- Singularity structure of NLO calculations
- Subtraction schemes

2 Nagy Soper subtraction scheme

- General setup
- Applications
- Matching with Parton showers

3 Summary and Outlook

Introduction (and pre-summary)

- era of LHC: QCD-governed processes, large NLO contributions
- want fully differential distributions:
need stable + fast NLO Monte Carlo Event generators
- full description \Rightarrow combination w parton showers necessary

Motivation for new scheme

- number of phase space mappings greatly reduced
 \Rightarrow especially important for large number of external particles
- dipoles: derived from splitting functions of
"Parton shower with quantum interference" (Nagy, Soper, 2007)
 \Rightarrow once shower and scheme are implemented:
facilitates matching with NLO calculations
- here: concentrate on subtraction scheme

NLO corrections: general structure

Masterformula

for m particles in the final state

$$\sigma_{\text{NLO,tot}} = \sigma_{\text{LO}} + \sigma_{\text{NLO}},$$

$$\sigma_{\text{LO}} = \int d\Gamma_m |\mathcal{M}_{\text{Born}}^{(m)}|^2(s) \quad \text{leading order contribution}$$

$$\sigma_{\text{NLO}} = \sigma_{\text{real}} + \sigma_{\text{virt}},$$

$$\sigma_{\text{real}} = \int d\Gamma_{m+1} |\mathcal{M}^{(m+1)}|^2 \quad \text{real emission}$$

$$\sigma_{\text{virt}} = \int d\Gamma_m 2 \operatorname{Re}(\mathcal{M}_{\text{Born}}^{(m)} (\mathcal{M}_{\text{virt}}^{(m)})^*) \quad \text{virtual contribution}$$

with $d\Gamma$: phase space integral, \mathcal{M} matrix elements
(here: flux factors etc implicit)

Infrared divergencies in NLO corrections

- source of infrared divergence: integration over phase space of emitted massless particles in real and virtual contribution
(poles cancel in $\sigma_{\text{real}} + \sigma_{\text{virt}}$)
- appear in matrix elements as terms $\frac{1}{p_i p_j} = \frac{1}{E_i E_j (1 - \cos \theta_{ij})}$
 $E_j \rightarrow 0$: soft divergence, $\cos \theta_{ij} \rightarrow 1$: collinear divergence
- poles arise from **integration** of phase space of p_j
- eg in QCD $\tilde{p}_i \rightarrow p_i + p_j$ (omitted color factors etc)

$$q \rightarrow q g : \propto \frac{1}{\varepsilon^2} + \frac{3}{2\varepsilon}, \quad g \rightarrow q \bar{q} : \propto -\frac{1}{3\varepsilon}$$

- important: **this behaviour is the same for all processes**

Subtraction schemes

Dipole subtraction: general idea

- know that pole structure always the same
- matrix element level: in the singular limits,

$$|\mathcal{M}^{(m+1)}|^2 \longrightarrow D_{ij}(p_i, p_j) |\mathcal{M}^{(m)}|^2, \quad D_{ij} \sim \frac{1}{p_i p_j} \quad (1)$$

- D_{ij} : **dipoles**, contain complete singularity structure
- also means that

$$\int d\Gamma_{m+1} \left(|\mathcal{M}^{(m+1)}|^2 - \sum_{ij} D_{ij} |\mathcal{M}^{(m)}|^2 \right) = \text{finite}$$

- **general idea of dipole subtraction:** make use of (1), shift singular parts from $m+1$ to m particle phase space
 ⇒ **need to have a good (analytical) parametrization of the singularity structure**

Subtraction schemes

Dipole subtraction for total cross sections

Master formula

$$\begin{aligned}\sigma &= \sigma^{LO} + \sigma^{NLO} \\ \sigma^{NLO} &= \int_{m+1} d\sigma^R + \int_m d\sigma^V + \int d\sigma^C \\ &= \int_{m+1} (d\sigma^R - d\sigma^A) + \int_m (d\sigma^{\tilde{A}} + d\sigma^V + d\sigma^C),\end{aligned}$$

⇒ effectively added "0"; both integrals finite

$$\begin{aligned}\sigma_m^{NLO}(s) &= \int_m \left\{ |\tilde{\mathcal{M}}_{\text{virt}}(s; \varepsilon)|^2 + \mathbb{I}(\varepsilon) |\mathcal{M}_{\text{Born}}(s)|^2 \right. \\ &\quad \left. + \int_0^1 dx (\mathbf{K}(x) + \mathbf{P}(x; \mu_F)) |\mathcal{M}_{\text{Born}}(x, s)|^2 \right\}\end{aligned}$$

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Subtraction schemes

Ingredient for subtraction schemes: momentum matching

- previous slide: add and subtract "0" in terms of

$$\int d\Gamma_m \tilde{F}_{\text{sing}} |\mathcal{M}_{\text{Born}}^{(m)}|^2 - \int d\Gamma_{m+1} F_{\text{sing}} |\mathcal{M}_{\text{Born}}^{(m)}|^2$$

- addition and subtraction takes place in different phase spaces

$$p_{\tilde{a}}^{(m)} = F(p_a^{(m+1)}, p_b^{(m+1)}, \dots)$$

This function is highly scheme dependent !!!

requirement: keep total energy/ momentum conserved:

$$\sum_m p_{\tilde{a}} \stackrel{!}{=} \sum_{m+1} p_a$$

(sum over outgoing particles only)

General setup

Nagy Soper subtraction scheme

Main motivation for new subtraction scheme

- basic idea: can use the splitting functions in the parton shower as dipole subtraction terms
 - ⇒ have same behaviour in singular limits
- "turn around" of idea suggested by Nagy, Soper (hep-ph/0503053): use Catani Seymour Dipoles for shower algorithm
- introduce new matching between m and $m + 1$ phase spaces
 - ⇒ leads to a much smaller number of subtraction terms
 - especially important for large number of external particles
 - ⇒ same dipoles in shower and subtraction scheme: facilitates matching with NLO calculations

Shifting momenta

Difference 1: Shifting momenta

- matching between m and $m+1$ particle spaces requires reshuffling of momenta
- for

$$p_{\text{mother}}^{(m)} = p_{\text{daughter}, 1}^{(m+1)} + p_{\text{daughter}, 2}^{(m+1)},$$

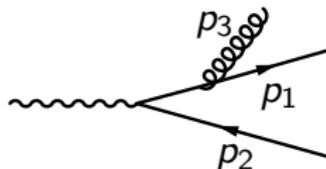
not all particles can be onshell simultaneously

- ⇒ need additional spectators to take over additional momenta
- Catani Seymour: define emitter-spectator pair, momentum goes to 1 additional particle only
- ⇒ quite easy integrations; however, for increasing number of particles, huge number of transformations necessary
- Nagy Soper:
 - shift momenta to **all** non-emitting external particles
- number of transformations = number of emitters
- leads to more complicated integrals during framework setup
- in general: # of transformations: $CS \sim N_{\text{jets}}^3/2$, $NS \sim N_{\text{jets}}^2/2$

Shifting momenta

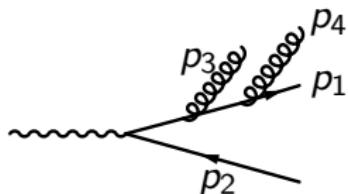
Shifting momenta: Example (1)

$$\gamma^* \longrightarrow q(p_1)\bar{q}(p_2)g(p_3) \text{ (@ NLO)}$$



part of Born contribution

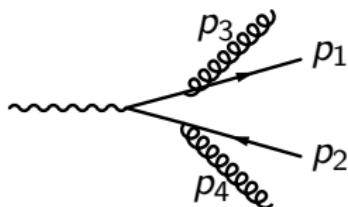
real gluon emissions for this diagramm:



CS: 1 momentum shift / spectator
 p_2, p_3 : 2 transformations
 NS: 1 total transformation

Shifting momenta

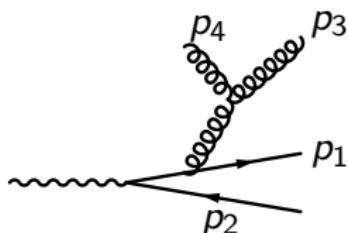
Shifting momenta: Example (2)



CS: 1 momentum shift / spectator

p_1 , p_3 : 2 transformations

NS: 1 total transformation



CS: 1 momentum shift / spectator

p_1 , p_2 : 2 transformations

NS: 1 total transformation

⇒ from simple counting:

12 transformations using CS vs 6 using NS dipoles !!

of course many more contributions (eg $g \rightarrow q \bar{q}$, other Born terms, ...)

Shifting momenta

Final state $g \rightarrow q \bar{q}$: Catani Seymour vs Nagy Soper (1)

- $g(\tilde{p}_i) \rightarrow q(p_i) + \bar{q}(p_j)$,
spectator: any other final state parton, p_k
- Dipole (in terms of integration variables):

$$D_{\text{NS, CS}}^{ij,k} \propto \underbrace{\frac{1}{y}}_{\text{sing}} \left[1 - \frac{z(1-z)}{1-\varepsilon} \right]$$

- NS definitions

$$y_{\text{NS}} = \frac{p_i p_j}{(p_i + p_j)Q - p_i p_j}, \quad z_{\text{NS}} = \frac{p_j \tilde{n}}{p_i \tilde{n} + p_j \tilde{n}}$$

$$\tilde{n} = \frac{1+y+\lambda}{2\lambda} Q - \frac{a}{\lambda} (p_i + p_j), \quad \lambda = \sqrt{(1+y)^2 - 4ay}, \quad a = \frac{Q^2}{(p_i + p_j)Q - p_i p_j}$$

- CS definitions:

$$y_{\text{CS}} = \frac{p_i p_j}{p_i p_j + p_i p_k + p_j p_k}, \quad z_{\text{CS}} = \frac{p_i p_k}{p_i p_k + p_j p_k}$$

Shifting momenta

Final state $g \rightarrow q \bar{q}$: Catani Seymour vs Nagy Soper (2)

- CS matching (all other final state particles untouched)

$$\tilde{p}_i = p_i + p_j - \frac{y}{1-y} p_k, \quad \tilde{p}_k^\mu = \frac{1}{1-y} p_k$$

- NS matching

$$\tilde{p}_i = \frac{1}{\lambda} (p_i + p_j) - \frac{1 - \lambda + y}{2 \lambda a} Q, \quad \tilde{p}_k^\mu = \Lambda^\mu{}_\nu p_k^\nu \quad \text{all fs particles}$$

$$\Lambda^{\mu\nu} = g^{\mu\nu} - \frac{2(K + \tilde{K})^\mu (K + \tilde{K})^\nu}{(K + \tilde{K})^2} + \frac{2K^\mu \tilde{K}^\nu}{K^2}, \quad K = Q - p_i - p_j, \quad \tilde{K} = Q - \tilde{p}_i$$

- integration measure (identical, same pole structure)

$$[dp_j]_{\text{CS}} = \frac{(2 \tilde{p}_i \tilde{p}_k)^{1-\varepsilon}}{16 \pi^2} \frac{d\Omega_{d-3}}{(2\pi)^{1-\varepsilon}} dz dy (1-y)^{1-2\varepsilon} y^{-\varepsilon} [z(1-z)]^{-\varepsilon},$$

$$[dp_j]_{\text{NS}} = \frac{(2 \tilde{p}_i Q)^{1-\varepsilon}}{16 \pi^2} \frac{d\Omega_{d-3}}{(2\pi)^{1-\varepsilon}} dz dy \lambda^{1-2\varepsilon} y^{-\varepsilon} [z(1-z)]^{-\varepsilon}$$

Shifting momenta

Final state $g \rightarrow q \bar{q}$: Catani Seymour vs Nagy Soper (3)

- result CS

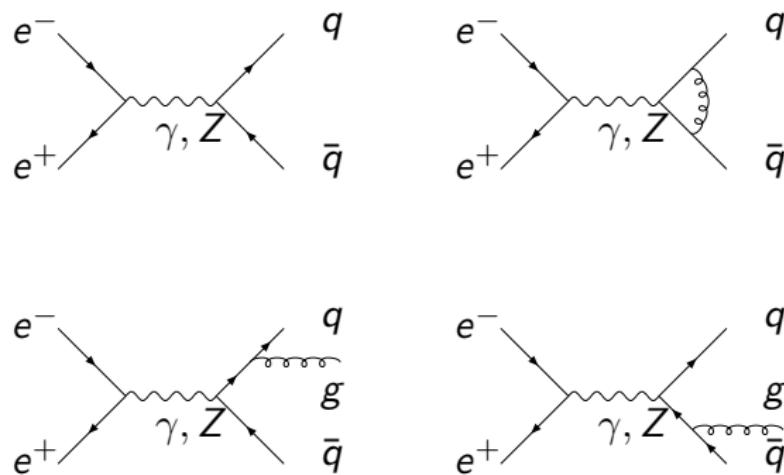
$$\mu^{2\varepsilon} \int [dp_j] D^{ij,k} = \frac{\alpha_s}{2\pi} \frac{1}{\Gamma(1-\varepsilon)} T_R \left(\frac{2\mu^2\pi}{\tilde{p}_i \tilde{p}_k} \right)^\varepsilon \left[-\frac{2}{3\varepsilon} - \frac{16}{9} \right]$$

- result NS

$$\begin{aligned} \mu^{2\varepsilon} \int [dp_j] D^{ij} &= T_R \frac{\alpha_s}{2\pi} \frac{\alpha_s}{\Gamma(1-\varepsilon)} \left(\frac{2\pi\mu^2}{p_i Q} \right)^\varepsilon \\ &\times \left[-\frac{2}{3\varepsilon} - \frac{16}{9} + \frac{2}{3} [(a-1) \ln(a-1) - a \ln a] \right], \end{aligned}$$

- for $a = 1$, reduces completely to Catani Seymour result
- (reason: $a = 1$ implies only 2 particles in the final state, $\tilde{n} \rightarrow p_k$, \Rightarrow complete equivalence)
- tradeoff: all final state particles get additional momenta: integral more complicated, but fewer transformations necessary

Applications

Applications: $e^+e^- \rightarrow 2 \text{ jets (1)}$ (slide by C.Chung)

Tree level diagram:
 $e^+e^- \rightarrow q(p_1) + \bar{q}(p_2)$

Virtual corrections:
 $e^+e^- \rightarrow q(p_1) + \bar{q}(p_2)$

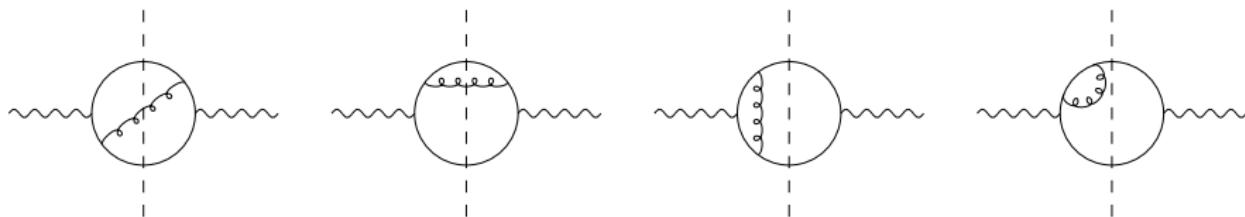
Real corrections:
 $e^+e^- \rightarrow$
 $q(p_1) + \bar{q}(p_2) + g(p_3)$

The matrix element for NLO real emission (three particle ps):

$$|\mathcal{M}_3(p_1, p_2, p_3)|^2 = C_F \frac{8\pi\alpha_s}{Q^2} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} |\mathcal{M}_2|^2, \quad x_i = \frac{2p_i \cdot Q}{Q^2}$$

($\mathcal{M}_2, \mathcal{M}_3$ averaged over angles)
 soft/ collinear singularities from $x_i \rightarrow 1$

Applications

Applications: $e^+e^- \rightarrow 2 \text{ jets (2)}$ (slide by C. Chung)

2 dipole contributions \mathcal{D}_1 and \mathcal{D}_2 (in 3 particle ps):

$$\begin{aligned} \mathcal{D}_1 &= v_{qgq}^2 - v_{\text{soft}}^2 = (v_{qgq}^2 - v_{\text{eik}}^2) + (v_{\text{eik}}^2 - v_{\text{soft}}^2) \\ &= \frac{4}{Q^2} \left\{ \left(\frac{1}{x_2} \right) \left[2 \left(\frac{x_1}{2-x_1-x_2} - \frac{1-x_2}{(2-x_1-x_2)^2} \right) + \frac{1-x_1}{1-x_2} \right] \right. \\ &\quad \left. + 2 \left(\frac{x_1+x_2-1}{1-x_2} \right) \frac{x_1}{(1-x_1)x_1+(1-x_2)x_2} \right\} \end{aligned}$$

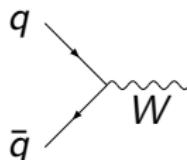
Integration over dipole

$$2 \left(\frac{4\pi\alpha_s}{2} \right) \mu^{2\epsilon} C_F \int d\zeta_p \mathcal{D}_1 = \frac{\alpha_s}{2\pi} C_F \frac{1}{\Gamma(1-\epsilon)} \left(\frac{4\pi\mu^2}{Q^2} \right)^\epsilon \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} - 2 + \frac{\pi^2}{3} \right)$$

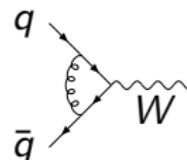
$$\sigma^{NLO} = \sigma^{NLO\{2\}} + \sigma^{NLO\{3\}} = \frac{3}{4} \frac{\alpha_s}{\pi} C_F \sigma^{LO}$$

Applications

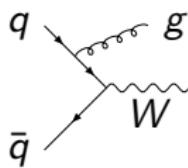
Single W production (slide by C. Chung)



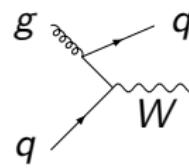
Tree level: $q\bar{q} \rightarrow W$



Virtual corrections: $q\bar{q} \rightarrow W$



Real corrections: $q\bar{q} \rightarrow Wg$



$gq \rightarrow Wq$ (+ 2 more diagrams)

$$\frac{1}{4} \frac{1}{9} |\mathcal{M}_B|^2 = \frac{g^2}{12} |V_{qq'}|^2 M_W^2, \quad \frac{1}{4} \frac{1}{9} \sum |\mathcal{M}_R|^2 = \frac{8g^2\pi\alpha_s}{9} |V_{qq'}|^2 \frac{\hat{t}^2 + \hat{u}^2 + 2M_W^2 \hat{s}}{\hat{t}\hat{u}}$$

$$|\mathcal{M}_V|^2 = |\mathcal{M}_B|^2 \frac{\alpha_s}{2\pi} C_F \frac{1}{\Gamma(1-\epsilon)} \left(\frac{4\pi\mu^2}{Q^2} \right)^\epsilon \left\{ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2 + \mathcal{O}(\epsilon) \right\}$$

Subtraction terms à la Nagy Soper

- 2 particle phase space (real emission)

$$\mathcal{D}^{14,2} + \mathcal{D}^{24,1} = \frac{8}{9} \pi \alpha_s g^2 \left(\frac{t^2 + u^2 + 2s p_3^2}{tu} \right) = \underbrace{\frac{1}{4} \frac{1}{9} \sum}_{\text{singular}} |\mathcal{M}_{\text{real}}|^2$$

- 1 particle phase space (virtual contribution)

$$\mathbf{I}(\epsilon) |\mathcal{M}_b|^2 = \underbrace{\frac{2\alpha_s}{3\pi} \frac{1}{\Gamma(1-\epsilon)} (-8 + \frac{2}{3}\pi^2)}_{\text{finite}} |\mathcal{M}_b|^2 - \underbrace{|\widetilde{\mathcal{M}}_v|^2}_{\text{singular (+finite)}}$$

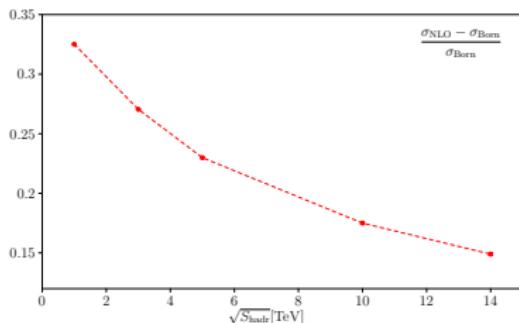
$$\begin{aligned} \mathbf{K}^a(xp_a) &= \frac{\alpha_s}{2\pi} C_F \frac{1}{\Gamma(1-\epsilon)} [- (1-x) \ln x + 2(1-x) \ln(1-x) \\ &\quad + 4x \left(\frac{\ln 1-x}{1-x} \right)_+ - \frac{2x \ln x}{(1-x)_+} - \left(\frac{1+x^2}{1-x} \right)_+ \ln \left(\frac{4\pi\mu^2}{2xp_a \cdot p_b} \right)] \end{aligned}$$

$$\mathbf{P}(x, \mu_F^2) = \frac{\alpha_s}{2\pi} C_F \frac{1}{\Gamma(1-\epsilon)} \left(\frac{1+x^2}{1-x} \right)_+ \ln \left(\frac{4\pi\mu^2}{\mu_F^2} \right)$$

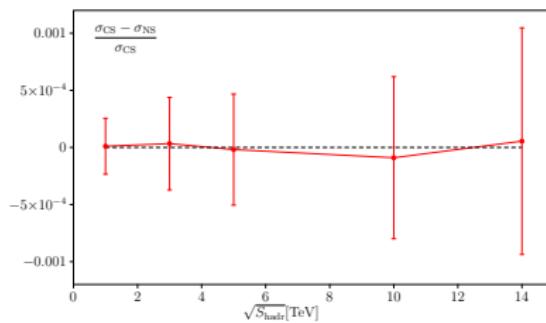
Applications

Numerical results for single W (slide by C. Chung)

input: $M_W = 80.35$ GeV, PDF \Rightarrow cteq6m, $\alpha_s(M_W) = 0.120299$



$\frac{\sigma_{NLO} - \sigma_{LO}}{\sigma_{LO}}$ as a function of $\sqrt{S_{\text{hadr}}}$
corrections up to 30%



relative difference between CS and NS: $\frac{\sigma_{CS} - \sigma_{NS}}{\sigma_{CS}}$
agree on the sub-permill level ✓

Matching with Parton showers

Difference 2: Matching with parton showers

- double counting: hard real emissions are described in both shower and "real emission" matrix element
- avoid double counting

$$-\int_{m+1} d\sigma^{\text{PS}}|_{m+1} + \int_{m+1} d\sigma^{\text{PS}}|_m$$

details eg in hep-ph/0204244: "Matching NLO QCD computations and parton shower simulations" (Frixione, Webber), MC@NLO

- important: have new terms in $m + 1$ phase space

$$\int_{m+1} \left(d\sigma^R - \underbrace{d\sigma^A}_{=0} + \underbrace{d\sigma^{\text{PS}}|_m}_{=0} - d\sigma^{\text{PS}}|_{m+1} \right)$$

- same splitting functions: second and third term cancel analytically !!
- ⇒ improves numerical efficiency

Status quo (instead of Summary)

- goal: establish NS dipole formalism
- all integrals are done ✓
- all singularity structures checked ✓
- finite term cross checks: almost done (missing: processes w more than 2 partons in the final state, initial state final state interference terms)

Checked processes

- single W at hadron colliders
- Dijet production at lepton colliders
- $p\bar{p} \rightarrow H$ and $H \rightarrow gg$
- DIS and $e e \rightarrow 3$ jets:
singularity cancellation checked, rest underway

Outlook

Outlook

- continue checks by application to simple processes for unchecked splitting functions
(initial final interference, $m > 2$ in final state)
- implement on matrix element level
- match with parton shower (Z. Nagy; underway)
- apply in (new) higher order calculations
- (more to come)

! Thanks for listening !

Appendix

More on counting

Maximal number of transformations

emitter, spectator	CS	NS
fin,fin	$N'(N'-1)(N'-2)/2$	$N'(N'-1)/2$
fin,ini	$N'(N'-1)$	—
ini,fin	$2(N'-1)N'$	$2N'$
ini,ini	$2N'$	—
total	$N'^2(N'+3)/2 = (N+1)^2(N+4)/2$	$N'(N'+3)/2 = (N+1)(N+4)/2$

Table: Maximal number of **transformations** needed for N particles in the born final state ($N' = N + 1$ in the real radiation contribution) using Catani Seymour or Nagy Soper prescriptions. Formula is exact for processes where all partons are gluons; for quarks, number can actually be smaller. Note: number of transformations is **not** equal to number of dipoles.

Real formulas

Dipole subtraction: Real master formula

Real Masterformula ($s = (p_a + p_b)^2$)

$$\begin{aligned} \sigma(s) &= \int_m d\Phi^{(m)}(s) \frac{1}{n_c(a)n_c(b)} |\mathcal{M}^{(m)}|^2(s) F_J^{(m)} \\ &+ \int d\Phi^{(m+1)}(s) \left\{ \frac{1}{n_c(a)n_c(b)} |\mathcal{M}^{(m+1)}|^2(s) F_J^{(m+1)} - \sum_{\text{dipoles}} (\mathcal{D} \cdot F_J^{(m)}) \right\} \\ &+ \int d\Phi^{(m)}(s) \left\{ \frac{1}{n_c(a)n_c(b)} |\mathcal{M}^{(m)}|_{\text{loop}}^2(p_a, p_b) + \mathbf{I}(\varepsilon) |\mathcal{M}^{(m)}|^2(s) \right\}_{\varepsilon=0} F_J^{(m)} \\ &+ \left\{ \int dx_a dx_b \delta(x - x_a) \delta(x_b - 1) \int d\Phi^{(m)}(x_a p_a, x_b p_b) |\mathcal{M}^{(m)}|^2(x_a p_a, x_b p_b) \right. \\ &\times \left. \left(\mathbf{K}^{a,a'}(x) + \mathbf{P}^{a,a'}(x_a p_a, x_b p_b, x; \mu_F^2) \right) \right\} + (a \leftrightarrow b) \end{aligned}$$

where all colour/ phase space factors have been accounted for

Second ingredient: Parametrization of integration variables

- again: remember you have

$$\begin{aligned} F_{\text{sing}} &\propto D_{ij}, \quad \tilde{F}_{\text{sing}} = \int d\Gamma_1 D_{ij}, \quad d\Gamma_1 \propto d^4 p_j \delta(p_j^2) \\ \implies \tilde{F}_{\text{sing}} &\propto \int d^4 p_j \delta(p_j^2) D_{ij} \end{aligned}$$

- 3 free variables (in D dimensions: $D - 1$)
!! need to be written in terms of m particle variables !!
- now all ingredients:
total energy momentum conservation, onshellness of external particles, need for integration variables

Integrated Dipoles in more details: I, K, P (1)

$m + 1$ phase space: in principle easy

$$\int d\Gamma_{m+1} \left(|\mathcal{M}_{\text{real}}|^2 - \sum D \right), \text{ finite}$$

m particle phase space: more complicated

need integration variables (emission from p_1):

$$x = 1 - \frac{p_4(p_1 + p_2)}{p_1 p_2} \text{ softness, } \tilde{v} = \frac{p_1 p_4}{p_1 p_2} \text{ collinearity}$$

Integrated Dipoles in more details: I, K, P (2)

- in principle, obtain $\int d\Gamma_1 D = \int_0^1 dx \left(\mathbf{I}(\varepsilon) + \tilde{\mathbf{K}}(x, \varepsilon) \right)$
- $\mathbf{I}(\varepsilon) \propto \delta(1 - x)$: corresponds to loop part
- $\tilde{\mathbf{K}}(x, \varepsilon)$ contains finite parts as well as **collinear singularities**
- latter need to be cancelled by adding **collinear counterterm**

$$\frac{1}{\epsilon} \left(\frac{4\pi\mu^2}{\mu_F^2} \right)^\epsilon P^{qq}(x)$$

depends on factorization scale μ_F ($P^{qq}(x)$ splitting function)

- PDFs come in again: term already accounted for by folding w PDF, needs to be subtracted
- for $q g \rightarrow W q$ like processes, only singularity which appears

$q \rightarrow q g$ for initial state quarks: Catani Seymour (1)

- $q(\tilde{p}_1) \rightarrow q(p_1) + g(p_4)$, q enters hard interaction
- Dipole:

$$D^{14,2} = -\frac{8\pi\mu^2\alpha_s C_F}{s+t+u} \left(\frac{2s(s+t+u)}{t(t+u)} + (1-\varepsilon) \frac{t+u}{t} \right)$$

- matching ($\tilde{p}_2 = p_2$)

$$\tilde{p}_1 = x p_1, \quad x = 1 - \frac{p_4(p_1 + p_2)}{(p_1 p_2)}$$

$$\tilde{p}_k^\mu = \Lambda^\mu{}_\nu p_k^\nu, \quad (k: \text{final state particles})$$

$$\Lambda^{\mu\nu} = -g^{\mu\nu} - \frac{2(K + \tilde{K})^\mu(K + \tilde{K})^\nu}{(K + \tilde{K})^2} + \frac{2K^\mu\tilde{K}^\nu}{K^2}$$

$$K = p_1 + p_2 - p_4, \quad \tilde{K} = \tilde{p}_1 + p_2$$

$q \rightarrow q g$ for initial state quarks: Catani Seymour (2)

- integration variables:

$$\nu = \frac{p_1 p_4}{p_1 p_2}, \quad x = 1 - \frac{p_4 (p_1 + p_2)}{(p_1 p_2)}$$

- in p_1, p_2 cm system: $E_4 \rightarrow 0 \Rightarrow x \rightarrow 1$ (softness)
 $\cos \theta_{14} \rightarrow 1 \Rightarrow \nu \rightarrow 0$ (collinearity)
- Dipole in terms of integration variables

$$D^{14,2} = -\frac{8 \pi \alpha_s C_F}{\nu x s} \left(\frac{1+x^2}{1-x} - \varepsilon(1-x) \right)$$

- integration measure

$$[dp_j] = \frac{(2 p_1 p_2)^{1-\varepsilon}}{16 \pi^2} \frac{d\Omega_{d-3}}{(2 \pi)^{1-\varepsilon}} dv dx (1-x)^{-2\varepsilon} \left[\frac{\nu}{1-x} \left(1 - \frac{\nu}{1-x} \right) \right]^{-\varepsilon}$$

where $\nu \leq 1 - x$ and all integrals between 0 and 1

Real formulas

$q \rightarrow q g$ for initial state quarks: Catani Seymour (3)

- result

$$\begin{aligned} \mu^{2\varepsilon} \int [dp_j] D^{14,2} &= \frac{\alpha_s}{2\pi} \frac{1}{\Gamma(1-\varepsilon)} C_F \left(\frac{2\mu^2\pi}{p_1 p_2} \right)^\varepsilon \\ &\times \int_0^1 dx \left(\mathbf{I}(\varepsilon) \delta(1-x) + \tilde{\mathbf{K}}(x, \varepsilon) \underbrace{- \frac{1}{\varepsilon} P^{qq}(x)}_{\text{killed by coll CT}} \right) \end{aligned}$$

with

$$\mathbf{I}(\varepsilon) = \frac{1}{\varepsilon^2} + \frac{3}{2\varepsilon} - \frac{\pi^2}{6}$$

$$\mathbf{K}(x) = (1-x) - 2(1+x) \ln(1-x) + \left(\frac{4}{1-x} \ln(1-x) \right)_+$$

$$P^{qq}(x) = \left(\frac{1+x^2}{1-x} \right)_+ \text{ regularized splitting function}$$

$q \rightarrow q g$ for initial state quarks: Nagy Soper (1)

- $q(\tilde{p}_1) \rightarrow q(p_1) + g(p_4)$, q enters hard interaction
- Dipole:

$$D^{14,2} = -\frac{8\pi\mu^2\alpha_s C_F}{s+t+u} \left(\frac{2su(s+t+u)}{t(t^2+u^2)} + (1-\varepsilon) \frac{u}{t} \right)$$

as CS, same pole structure as CS

- matching, integration variables, integration measure:
as Catani Seymour($v \leftrightarrow y$)
- Dipole in terms of integration variables

$$\begin{aligned} D^{14,2} &= -\frac{8\pi\alpha_s C_F}{xs} \\ &\times \left(\frac{1-x-y}{y}(1-\varepsilon) + \frac{2x}{y(1-x)} - \frac{2x[2y-(1-x)]}{(1-x)[y^2+(1-x-y)^2]} \right) \end{aligned}$$

Real formulas

$q \rightarrow q g$ for initial state quarks: Nagy Soper (2)

- result

$$\begin{aligned} \mu^{2\varepsilon} \int [dp_j] D^{14,2} &= \frac{\alpha_s}{2\pi} \frac{1}{\Gamma(1-\varepsilon)} C_F \left(\frac{2\mu^2\pi}{p_1 p_2} \right)^\varepsilon \\ &\times \int_0^1 dx \left(\textcolor{red}{I(\varepsilon)} \delta(1-x) + \textcolor{green}{\tilde{K}(x, \varepsilon)} \underbrace{-\frac{1}{\varepsilon} P^{qq}(x)}_{\text{killed by coll CT}} \right) \end{aligned}$$

with

$$\textcolor{red}{K(x)} =$$

$$(1-x) - 2(1+x) \ln(1-x) + \left(\frac{4}{1-x} \ln(1-x) \right)_+ -(1-x)$$

- equivalence of dipoles schemes checked analytically